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Certain Partial Differential Equations

Connected

With the Theory of Surfaces.

Dissertation
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By

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Introduction.

In two notes in the Comptes Rendus of Oct. 26, 1896, and Nov. 16, 1896, and in the American Journal of Mathematics of Jan., 1897, Professor Craig has considered certain partial differential equations connected with the theory of surfaces.

In this paper the work of these three has been continued. The first section is introductory, and gives a brief resumé of Laplace's method for the integration of linear partial differential equations of the second order.

In the next two sections

Professor Craig's equations ($E_{0,1}$) and ($E_{0,2}$) are deduced, together with their equivalent equations and the corresponding Laplace's series, and the general formulae for calculating their invariants are derived. The form of substitution required to pass from an equation of one series to one of the other series has also been found.

In the fourth section, it is shown that by making the two equations ($E_{0,1}$) and ($E_{0,2}$) identical the linear self-conjugate form and its canonical system and the reciprocals of the two principal

radii of curvature satisfy adjoint equations.

The next two sections are taken up with the calculation of coefficients of the general transformed equations and making the general equations equal. Here the linear element reduces to the isothermal form.

In the seventh section the hypothesis $b_{01} = a_{02} = 0$. has been made. Here the equations (E_{01}) and (E_{02}) reduce to integrable forms and the lines of curvature form an isothermal system.

The following section has been devoted to the consideration

of the hypothesis

$$a_{01} = b_{02},$$
$$a_{02} = b_{01}.$$

Then the conditions have been found that must exist in order that the reciprocals of the principal radii of curvature of parallel surfaces may satisfy the same partial differential equations as those of the original surface.

The forms to which the equations (Eo1) and (Eo2) reduce for the ellipsoid and the cyclide of Dupin are then given and it is seen that they may be readily integrated by the method considered by Poisson.

I desire to make here

acknowledgment of gratitude to Professor Ewing, who suggested to me the subject of this dissertation, and to whom I am indebted for valuable assistance

1. Consider the linear partial differential equation

$$(1) \qquad \frac{\partial^2 \varphi}{\partial u \partial v} + a\frac{\partial \varphi}{\partial u} + b\frac{\partial \varphi}{\partial v} + c\varphi = 0,$$

where a, b, c are any functions whatever of u and v; it can be put in either of the two following forms:

$$(2) \quad \begin{cases} \frac{\partial}{\partial u}\left(\frac{\partial \varphi}{\partial v} + a\varphi\right) + b\frac{\partial \varphi}{\partial v} + \left(c - \frac{\partial a}{\partial u}\right)\varphi = 0, \\ \frac{\partial}{\partial v}\left(\frac{\partial \varphi}{\partial u} + b\varphi\right) + a\frac{\partial \varphi}{\partial u} + \left(c - \frac{\partial b}{\partial v}\right)\varphi = 0. \end{cases}$$

Introduce the two functions h and k defined as follows:

$$(3) \quad \begin{cases} h = \frac{\partial a}{\partial u} + ab - c \\ k = \frac{\partial b}{\partial v} + ab - c. \end{cases}$$

If h is zero, the first of equations (2) can be written

$$\frac{\partial}{\partial u}\left(\frac{\partial \varphi}{\partial v} + a\varphi\right) + b\left(\frac{\partial \varphi}{\partial v} + a\varphi\right) = 0.$$

Therefore, the complete determination of all the integrals of equation (1) is reduced to the successive integration of two equations of the first order

$$(4)\quad \begin{cases} \dfrac{\partial \varphi_1}{\partial u} + b\varphi_1 = 0, \\ \dfrac{\partial \varphi}{\partial v} + a\varphi = \varphi_1, \end{cases}$$

integration which requires only quadratures.

In like manner, if k is zero, the second of equations (2) can be put in the form

$$\frac{\partial}{\partial v}\left(\frac{\partial \varphi}{\partial u} + b\varphi\right) + a\left(\frac{\partial \varphi}{\partial u} + b\varphi\right) = 0,$$

and we can replace equation (1) by the two equations

$$(5)\quad \begin{cases} \dfrac{\partial \varphi_1}{\partial v} + a\varphi_1 = 0, \\ \dfrac{\partial \varphi}{\partial u} + b\varphi = \varphi_1. \end{cases}$$

Suppose now that h and k are not zero. We shall show that these functions enjoy properties of invariance, which have a great importance in the study of the proposed equation. Consider successively the substitutions

$$\varphi = \lambda \varphi';$$

$$u = f(u'), \quad v = \psi(v');$$

$$u = v', \quad v = u'.$$

Make at first

$$(6) \qquad \varphi = \lambda \varphi';$$

λ being any function of u and v. The equation in φ will become

$$(7) \quad \left\{ \begin{array}{l} \dfrac{\partial^2 \varphi'}{\partial u \partial v} + \left(a + \dfrac{\partial \log \lambda}{\partial v}\right) \dfrac{\partial \varphi'}{\partial u} + \left(b + \dfrac{\partial \log \lambda}{\partial u}\right) \dfrac{\partial \varphi'}{\partial v} \\ + \left(c + a \dfrac{\partial \log \lambda}{\partial u} + b \dfrac{\partial \log \lambda}{\partial v} + \dfrac{1}{\lambda} \dfrac{\partial^2 \lambda}{\partial u \partial v}\right) \varphi' = 0. \end{array} \right.$$

If we calculate the new values of h and k, we find that these functions have not changed.

Make now the substitution

$$u = f(u'), \quad v = \varphi(v').$$

Equation (1) will become

$$\frac{\partial^2 \theta}{\partial u' \partial v'} + a\psi'(v')\frac{\partial \theta}{\partial u'} + b f'(u')\frac{\partial \theta}{\partial v'} + c f'(u')\psi'(v')\theta = 0$$

The new values h', k' of h and k will be

$$(8) \quad \begin{cases} h' = h f'(u')\varphi'(v'), \\ k' = k f'(u')\varphi'(v'). \end{cases}$$

Finally, the substitution

$$u = v', \quad v = u'$$

will exchange the values of h and k. All these properties justify the name of invariants which we shall give to h and k.

Suppose that the invariants h and k are not zero. Make the substitution given by the equation

$$(9) \qquad \varphi_1 = \frac{\partial \varphi}{\partial v} + a\varphi.$$

Equation (1) can now be written

$$(10) \qquad \frac{\partial \varphi_1}{\partial u} + b\varphi_1 = h\varphi.$$

If we eliminate φ between these two equations we get

$$(11) \qquad \frac{\partial^2 \varphi_1}{\partial u \partial v} + a_1 \frac{\partial \varphi_1}{\partial u} + b_1 \frac{\partial \varphi_1}{\partial v} + c_1 \varphi_1 = 0,$$

where

$$(12) \qquad \begin{cases} a_1 = a - \dfrac{\partial \log h}{\partial v}, \quad b_1 = b, \\ c_1 = c - \dfrac{\partial a}{\partial u} + \dfrac{\partial b}{\partial v} - b\dfrac{\partial \log h}{\partial v}. \end{cases}$$

The values of the invariants will be

$$(13) \qquad \begin{cases} h_1 = 2h - k - \dfrac{\partial^2 \log h}{\partial u \partial v}, \\ k_1 = h \end{cases}$$

They are expressed only as functions of the invariants of the given equation.

Make also the substitution.

$$(14) \qquad \varphi_{-1} = \frac{\partial \varphi}{\partial u} + b\varphi.$$

We shall have

$$(15) \qquad \frac{\partial \varphi_{-1}}{\partial v} + a\varphi_{-1} = k\varphi,$$

and we get the following equation in φ_{-1}

$$(16) \qquad \frac{\partial^2 \varphi_{-1}}{\partial u \partial v} + a_{-1}\frac{\partial \varphi_{-1}}{\partial u} + b_{-1}\frac{\partial \varphi_{-1}}{\partial v} + c_{-1}\varphi_{-1} = 0.$$

In this the coefficients have the values

$$(17) \qquad \begin{cases} a_{-1} = a, \quad b_{-1} = b - \frac{\partial \log k}{\partial u}, \\ c_{-1} = c - \frac{\partial b}{\partial v} + \frac{\partial a}{\partial u} - a\frac{\partial \log k}{\partial u}. \end{cases}$$

The values of the invariants will be

$$(18) \qquad \begin{cases} h_{-1} = k, \\ k_{-1} = 2k - h - \frac{\partial^2 \log k}{\partial u \partial v}. \end{cases}$$

Thus we deduce from the proposed equation (E) two new equations (E_1) and (E_{-1}). We can apply the same method to these two equations, but we will not get two new equations for each of them. The first of the two substitutions applied to the equation (E_1) will give us the proposed equation in which φ will be replaced by $\frac{1}{\varphi}$. The second substitution, applied to (E_{-1}), will bring us back to the proposed equation in which φ will be replaced by $\frac{1}{\varphi}$.

If then we regard as equivalent two equations which reduce to each other by changing

points P and which leave, therefore the same invariants, we see that the substitution of Laplace applied successively will give only one linear series of equations

$$\cdots, (E_{-2}), (E_{-1}), (E), (E_1), (E_2), \cdots$$

with positive and negative indices.

The invariants of equations (E_i) are deduced from each other by the repeated application of equations (13) and (18). We find thus

$$(19)\quad \begin{cases} h_{i+1} = 2h_i - k_i - \dfrac{\partial^2 \log h_i}{\partial u \partial v}, \\ k_{i+1} = h_i. \end{cases}$$

Solved with respect to h_i or k_i, these formulas give

$$(20)\quad \begin{cases} h_i = k_{i+1}, \\ k_i = 2k_{i+1} - h_{i+1} - \dfrac{\partial^2 \log k_{i+1}}{\partial u \partial v} \end{cases}$$

We can give to i all integer values, positive or negative. Instead of considering two series of quantities h_i and k_i we could introduce only the quantities h_i. We shall then have the series

$$\ldots\ldots, h_{-2}, h_{-1}, h_0, h_1, h_2, \ldots\ldots,$$

reducing one from the other by the formula

$$(2.1) \qquad h_{i+1} + h_{i-1} = 2h_i - \frac{\partial^2 \log h_i}{\partial u \partial v}.$$

The invariants of the equation (E_i) will be h_i and h_{i-1}. By the linear combination of equations (2.1) we can obtain the relation

$$(2.2) \qquad h_{i+1} = h_i + h - k - \frac{\partial^2 \log h h_1 \ldots h_i}{\partial u \partial v}.$$

2. Let u and v denote the parameters of the lines of curvature of a surface, ρ_1 and ρ_2 the principal radii of curvature of the surface at the point (u, v), ρ_1 being the principal radius corresponding to the line $v = \text{const.}$ (the u-line) and ρ_2 corresponding to $u = \text{const.}$ (the v line). Let R_1, R_2 denote the radii of geodesic curvature of the lines $u = \text{const.}$ and $v = \text{const.}$ respectively.

We have now

$$\rho_1 = \frac{E}{L}, \quad \rho_2 = \frac{G}{N},$$

$$(23) \quad \begin{cases} \dfrac{1}{R_2} = \dfrac{-1}{2E\sqrt{EG}} \dfrac{\partial E}{\partial v}, \\ \dfrac{1}{R_1} = \dfrac{-1}{2G\sqrt{EG}} \dfrac{\partial G}{\partial u}. \end{cases}$$

We have also the following equations connecting all of the preceding quantities

$$(2.4)\quad \begin{cases} \frac{1}{\rho_1^2}\frac{\partial \rho_1}{\partial v} + \frac{\sqrt{E_2}}{R_2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = 0, \\ \frac{1}{\rho_2^2}\frac{\partial \rho_2}{\partial u} - \frac{\sqrt{E_1}}{R_1}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = 0. \end{cases}$$

Liouville's formula for the measure of curvature can be written

$$(2.5)\quad \frac{\sqrt{E_1 E_2}}{\rho_1 \rho_2} = \frac{\partial}{\partial u}\frac{\sqrt{E_2}}{\rho_1} + \frac{\partial}{\partial v}\frac{\sqrt{E_1}}{\rho_2}.$$

Substituting the values of ρ_1 and ρ_2 in equations (2.3), they can be reduced to the form

$$(2.6)\quad \begin{cases} \frac{\partial}{\partial v}\frac{\sqrt{E_1}}{\rho_1} = -\frac{\sqrt{E_1}}{R_2}\frac{\sqrt{E_2}}{\rho_2}, \\ \frac{\partial}{\partial u}\frac{\sqrt{E_2}}{\rho_2} = -\frac{\sqrt{E_2}}{R_1}\frac{\sqrt{E_1}}{\rho_1}. \end{cases}$$

Differentiating the first of

equations (24) for u and v, the second for v and eliminating $\frac{1}{\rho_2}$ and $\frac{1}{\rho_1}$ respectively, we obtain the equations

$$(27)\begin{cases} \frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_1} - \frac{\sqrt{G}}{R_2}\frac{\partial}{\partial u}\frac{1}{\rho_1} - \left(\frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} + \frac{\sqrt{E}}{R_1}\right)\frac{\partial}{\partial v}\frac{1}{\rho_1} = 0, \\ \frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_2} - \left(\frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1} + \frac{\sqrt{G}}{R_2}\right)\frac{\partial}{\partial u}\frac{1}{\rho_2} - \frac{\sqrt{E}}{R_1}\frac{\partial}{\partial v}\frac{1}{\rho_2} = 0. \end{cases}$$

That is, $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ are particular integrals of the differential equations

$$(28)\begin{cases} (\delta_{01}) = \frac{\partial^2 \varphi_{01}}{\partial u \partial v} - \frac{\sqrt{G}}{R_2}\frac{\partial \varphi_{01}}{\partial u} - \left(\frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} + \frac{\sqrt{E}}{R_1}\right)\frac{\partial \varphi_{01}}{\partial v} = 0, \\ (\delta_{02}) = \frac{\partial^2 \varphi_{02}}{\partial u \partial v} - \left(\frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1} + \frac{\sqrt{G}}{R_2}\right)\frac{\partial \varphi_{02}}{\partial u} - \frac{\sqrt{E}}{R_1}\frac{\partial \varphi_{02}}{\partial v} = 0. \end{cases}$$

The differential equation satisfied by the coordinates (x, y, z) of the point (u, v) of the surface may now be written

$$(29)\qquad \frac{\partial^2\varphi_0}{\partial u\,\partial v}+\frac{\sqrt{G}}{R_2}\frac{\partial\varphi_0}{\partial u}+\frac{\sqrt{E}}{R_1}\frac{\partial\varphi_0}{\partial v}=0.$$

From equations (26) we derive in the same way as above the following:

$$(30)\qquad \begin{cases} \dfrac{\partial^2\varphi_{01}}{\partial u\,\partial v}-\dfrac{\partial}{\partial u}\log\dfrac{\sqrt{E}}{R_2}\dfrac{\partial\varphi_{01}}{\partial v}-\dfrac{\sqrt{EG}}{R_1R_2}\varphi_{01}=0, \\ \dfrac{\partial^2\varphi_{02}}{\partial u\,\partial v}-\dfrac{\partial}{\partial v}\log\dfrac{\sqrt{G}}{R_1}\dfrac{\partial\varphi_{02}}{\partial u}-\dfrac{\sqrt{EG}}{R_1R_2}\varphi_{02}=0. \end{cases}$$

These equations have $\frac{\sqrt{E}}{\rho_1}$ and $\frac{\sqrt{G}}{\rho_2}$ respectively as contravariant integrals. They are equivalent to equations (28); that is, they have the same invariants.

3. We shall now consider Laplace's series of equations corresponding to (ε_{01}) and (ε_{02}),

$$(31)\quad \begin{cases} \cdots,(\varepsilon_{-i,1}),\cdots,(\varepsilon_{-1,1}),(\varepsilon_{0,1}),(\varepsilon_{1,1}),(\varepsilon_{2,1}),\cdots,(\varepsilon_{i,1}),\cdots \\ \cdots,(\varepsilon_{-i,2}),\cdots,(\varepsilon_{-1,2}),(\varepsilon_{0,2}),(\varepsilon_{1,2}),(\varepsilon_{2,2}),\cdots,(\varepsilon_{i,2}),\cdots \end{cases}$$

Using equations (31), we find the moments of $E_{0,1}$ are

$$(32)\quad \begin{cases} h_{01} = \dfrac{\sqrt{E_0 E_1}}{R_1 R_2}, \\ k_{01} = -\dfrac{\partial^2}{\partial\omega\partial v}\log\dfrac{\sqrt{E_0}}{R_2} + \dfrac{\sqrt{E_0 E_1}}{R_1 R_2}; \end{cases}$$

those of $(E_{n,1})$ are

$$(33)\quad \begin{cases} h_{n1} = -\dfrac{\partial^2}{\partial\omega\partial v}\log\dfrac{\sqrt{E_n}}{R_1} + \dfrac{\sqrt{E_n E_1}}{R_1 R_2}, \\ k_{n1} = \dfrac{\sqrt{E_n E_1}}{R_1 R_2}. \end{cases}$$

The application of formulæ (2, 2,) will give the general moments. Thus we find for all positive values of i:

$$(34)\quad \begin{cases} h_{i1} = h_{i-1,1} + \dfrac{\partial^2}{\partial\omega\partial v}\log\dfrac{\sqrt{E_i}}{R_i} - \dfrac{\partial^2}{\partial\omega\partial v}\log h_{01} h_{11} \cdots h_{i-1} \\ \quad = h_{i-1,1} - \dfrac{\partial^2}{\partial\omega\partial v}\log\dfrac{\sqrt{E_i}}{R_i} h_{01} h_{11} \cdots h_{i-1,1}, \\ h_{i2} = h_{i-1,2} - \dfrac{\partial^2}{\partial\omega\partial v}\log\dfrac{\sqrt{E_i}}{R_i} h_{02} h_{12} \cdots h_{i-1,2}. \end{cases}$$

$$(35)\quad \begin{cases} k_{i1} = h_{i-1,1}, \\ k_{i2} = h_{i-1,2}. \end{cases}$$

For negative values of i we have

$$(34)\quad \begin{cases} h_{-i1} = h_{-(i-1),1} - \dfrac{\partial^2}{\partial u \partial v} \log \dfrac{\sqrt{E}}{R_2} h_{-11} h_{-21} \cdots h_{-(i-1),1}, \\ h_{-i2} = h_{-(i-1),2} + \dfrac{\partial^2}{\partial u \partial v} \log \dfrac{\sqrt{G}}{R_1} - \dfrac{\partial^2}{\partial u \partial v} \log h_{-12} h_{-22} \cdots h_{-(i-1),2} \\ \quad = h_{-(i-1),2} - \dfrac{\partial^2}{\partial u \partial v} \log \dfrac{\sqrt{E}}{R_2} h_{-12} h_{-22} \cdots h_{-(i-1),2}. \end{cases}$$

$$(37)\quad \begin{cases} k_{-i1} = k_{-(i+1),1}, \\ k_{-i2} = k_{-(i+1),2}. \end{cases}$$

From the first of equations (34) it may be seen that

$$h_{11} = \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{E}}{R_1}.$$

Therefore,

$$(38)\quad \begin{cases} h_{11} = h_{02}, \\ k_{11} = h_{01} = k_{02}. \end{cases}$$

Consider the general invariants. Let us suppose

$$(39)\quad h_{i-1,1} = h_{i-2,2};$$

it may be readily seen that

$$(40)\qquad \begin{cases} h_{i,1} = h_{i-1,2}, \\ k_{i,1} = k_{i-1,2}. \end{cases}$$

But we have found equation (37) to be true for $i = 2$; then it holds for all values of i. Therefore, we have the same invariants in the two series of equations (2), and invariant in the first series being equal to the preceding invariant in the second series. Then the equations $(\bar{6}_{i,1})$ and $(\bar{6}_{i-1,2})$ are equivalent. Hence the form may be reduced to the second by replacing the function $f_{i,1}$ by $\lambda f_{i-1,2}$. We can determine λ by identifying the coefficients in the two equations. By repeated application of equations (2)

we find the general coefficients

$$(4.1)\quad \begin{cases} a_{i1} = a_{01} - \dfrac{\partial \log h_{01} h_{11} \cdots h_{i-1,1}}{\partial v}, \\ a_{i-1,2} = a_{02} - \dfrac{\partial \log h_{02} h_{12} \cdots h_{i-2,2}}{\partial v} \\ \qquad = a_{02} - \dfrac{\partial \log h_{12} h_{22} \cdots h_{i-1,1}}{\partial v}. \end{cases}$$

$$(4.2)\quad \begin{cases} f_{i1} = f_{01}, \\ f_{i-1,2} = f_{02}. \end{cases}$$

The formulas of identification which we may take from equation (7) will give

$$(4.3)\quad \begin{cases} a_{i-1,2} = a_{i1} + \dfrac{\partial \log k}{\partial v}, \\ f_{i-1,2} = f_{i1} + \dfrac{\partial \log k}{\partial u}. \end{cases}$$

Replacing $a_{i-1,2}$ and a_{i1} in the first equation by their values taken from (4.1), we get the equation

$$\frac{\partial \log k}{\partial v} = \frac{\partial}{\partial v} \log \frac{T}{R_1}.$$

After integration it may be

placed in the form

$$\log I = \log \frac{E_2}{R_{i-1}} + \log \pi. \tag{4.4}$$

By substituting the values of $b_{i-1,2}$ and $b_{i,1}$ taken from (4.2) the record of equations (4.3) may be rewritten in the form

$$\frac{\partial \log I}{\partial n} = \frac{\partial}{\partial n} \log \frac{E_2}{R_{i-1}}.$$

Hence,

$$\log I = \log \frac{E_2}{R_{i-1}} + \log V. \tag{4.5}$$

By comparing (4.4) and (4.5) it may be seen that π and V must be constants. Therefore, neglecting a constant factor, which would disappear after substitution, we may take

$$I = \frac{E_2}{R_{i-1}}.$$

Hence, if $\frac{\quad}{\quad}(E_{i,1})$ the formula

φ_{i1} be replaced by $\frac{\sqrt{E}}{R_2}\cdot x_{i-1,2}$, we shall find $(\bar{E}_{i-1,2})$.

In the same way, it may be shown that replacing $p_{i-1,2}$ in $(\bar{E}_{i-1,2})$ by $\frac{\sqrt{E}}{R_1}p_{(i-1),1}$ will give the equation $(\bar{E}_{(i-1),1})$.

4. We shall now consider the result of making the equations $(\bar{E}_{0,1})$ and $(\bar{E}_{0,2})$ identical. By examining the coefficients it is evident that we have the condition

$$(4b)\quad \begin{cases} \frac{\partial}{\partial u}\log\frac{\sqrt{E}}{R_2} = 0, \\ \frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1} = 0. \end{cases}$$

The integral of the first of these equations may be written $\frac{\sqrt{E}}{R_2} = V.$

Substituting the value of $\frac{\sqrt{G}}{\rho_2}$ from the first of (23), we have

$$-\frac{\partial}{\partial v}\log\sqrt{G} = V;$$

hence,

$$G = U_1 V_2. \tag{47}$$

From the second of equations (46) we get

$$\frac{\sqrt{G}}{R_1} = U.$$

This, combined with (23), gives

$$-\frac{\partial}{\partial u}\log\sqrt{G} = U;$$

we have then

$$G = U_2 V_1. \tag{48}$$

The linear element of the surface may now be written

$$ds^2 = U_2 V_2\left(\frac{U_1}{U_2}du^2 + \frac{V_1}{V_2}dv^2\right),$$

or

$$ds^2 = \lambda(U du^2 + V dv^2), \tag{49}$$

the U_i being functions of u alone and the V_i of v alone. Then the lines of curvature form an isothermal system.

The equation (20) becomes now the form

$$(50) \qquad \frac{\partial^2\varphi}{\partial u\,\partial v} - \frac{\sqrt{E}}{R_2}\frac{\partial\varphi}{\partial u} - \frac{\sqrt{G}}{R_1}\frac{\partial\varphi}{\partial v} = 0,$$

which admits $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ as particular integrals. It may also be written

$$\frac{\partial^2\varphi}{\partial u\,\partial v} + \frac{\partial \log\sqrt{E}}{\partial v}\frac{\partial\varphi}{\partial u} + \frac{\partial\log\sqrt{G}}{\partial u}\frac{\partial\varphi}{\partial v} = 0.$$

Since $E = \lambda T$ and $G = \lambda V$, we have

$$(51) \qquad \frac{\partial^2\varphi}{\partial u\,\partial v} - \frac{\partial\log\sqrt{T}}{\partial v}\frac{\partial\varphi}{\partial u} - \frac{\partial\log\sqrt{T}}{\partial u}\frac{\partial\varphi}{\partial v} = 0.$$

The equation satisfied by (x, y, z) of any point (u, v) of the surface is now

(52) $$\frac{\partial^2 \psi_0}{\partial u \partial v} + \frac{\partial \log \sqrt{\lambda}}{\partial v}\frac{\partial \psi_0}{\partial u} + \frac{\partial \log \sqrt{\lambda}}{\partial u}\frac{\partial \psi_0}{\partial v} = 0.$$

If we have a linear equation

$$\frac{\partial^2 \theta}{\partial u \partial v} + a\frac{\partial \theta}{\partial u} + b\frac{\partial \theta}{\partial v} + c\theta = 0,$$

its adjoint will be of the form

$$\frac{\partial^2 \theta}{\partial u \partial v} - a\frac{\partial \theta}{\partial u} - b\frac{\partial \theta}{\partial v} + \left(c - \frac{\partial a}{\partial u} - \frac{\partial b}{\partial v}\right)\theta = 0.$$

In equation (51) we have

$$a = -\frac{\partial \log \sqrt{\lambda}}{\partial v}, \quad b = -\frac{\partial \log \sqrt{\lambda}}{\partial u}, \quad c = 0,$$

Hence, $$c - \frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} = 2\frac{\partial^2 \log \sqrt{\lambda}}{\partial u \partial v} = \frac{\partial^2 \log \lambda}{\partial u \partial v} = 0,$$

since $\lambda = \nabla_2 \nabla_2$. Then it follows that (52) is the adjoint of (51). Therefore, if u and v are the parameters of lines of curvature of a surface and if the linear element can be reduced to the

isothermal form

$$ds^2 = \lambda(\Theta du^2 + \nabla dv^2), \quad (\lambda = \Theta_2 \nabla_2)$$

then the Codazzian coordinates (x, y, z) satisfy a differential equation which is the adjoint of that satisfied by the reciprocals of the principal radii of curvature.

The invariants of (57) are

$$(53) \quad h = k = \frac{1}{4}\frac{\partial \log \lambda}{\partial u}\frac{\partial \log \lambda}{\partial v} = \frac{1}{2}\frac{\Theta_2'}{\Theta_2}\frac{\nabla_2'}{\nabla_2}.$$

Therefore, $\frac{\partial^2 \log h}{\partial u\, \partial v} = 0.$

We can find

$$(54) \quad \Theta_{11} = \frac{\partial^2 \rho_{11}}{\partial u\, \partial v} - \frac{\partial \log \sqrt{\lambda h}}{\partial u}\frac{\partial \rho_{11}}{\partial v} - \frac{\partial \log \sqrt{\lambda}}{\partial u}\frac{\partial \rho_{11}}{\partial v} + \frac{\partial \log \sqrt{\lambda}}{\partial u}\frac{\partial \log h}{\partial v}\rho_{11} = 0.$$

The invariants of this equation

are

$$(5.5) \quad \begin{cases} h_1 = \frac{1}{2}\frac{\partial \log \frac{1}{R_1}}{\partial v} - \frac{\partial \log \rho}{\partial v} = h_1, \\ k_1 = h_1. \end{cases}$$

Then we have

$$(5.6) \quad h_1 - k_1 = h_1 - h_1 = k - k.$$

Then all the constants on the series will be equal. Therefore we are led to the consideration of only one equation (2_0).

5. Make now the equations (6_1) and (6_2) identical. From equations (5_1) we obtain

$$a_{11} = -\frac{\sqrt{E}}{R_1} - \frac{\partial \log k_{01}}{\partial v},$$

$$a_{12} = -\frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1} - \frac{\sqrt{E}}{R_2} - \frac{\partial \log k_{02}}{\partial v}.$$

Equating these coefficients, we have

$$(5.7) \quad \frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_2} = \frac{\partial \log k_{02}}{\partial v}.$$

Making $k_{11} = k_{12}$ gives us

$$\frac{\partial}{\partial u} \log \frac{\sqrt{E}}{R_2} = 0. \tag{58}$$

We have also

$$c_{11} = -\frac{\partial}{\partial u}\frac{\sqrt{E}}{R_2} - \frac{\partial}{\partial u}\log\frac{\sqrt{E}}{R_2} - \frac{\partial}{\partial v}\frac{\sqrt{E}}{R_1} + \frac{\partial}{\partial u}\log\frac{\sqrt{E}}{R_2} + \frac{\sqrt{E}}{R_2}\frac{\partial \log h_{01}}{\partial u},$$

$$c_{12} = \frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1} + \frac{\partial}{\partial u}\frac{\sqrt{E}}{R_2} - \frac{\partial}{\partial v}\frac{\sqrt{E}}{R_1} + \frac{\sqrt{E}}{R_1}\frac{\partial \log h_{02}}{\partial u}.$$

By separating these values and taking (57) into account we get

$$\frac{\partial^2 \log h_{01}}{\partial u \partial v} = \frac{\partial}{\partial v}\frac{\sqrt{E}}{R_1}. \tag{59}$$

But from (23) we see that

$$\frac{\partial}{\partial v}\frac{\sqrt{E}}{R_1} = -\frac{\partial^2}{\partial u \partial v}\log\sqrt{G}.$$

Hence,

$$\frac{\partial^2}{\partial u \partial v}\log\sqrt{G}\,h_{01} = 0. \tag{60}$$

The integral of (58) may be written

$$\frac{\sqrt{E}}{R_2} = V,$$

and this combined with (23) gives

$$-\frac{\partial}{\partial v}\log\sqrt{E} = V'$$

Therefore by integrating this we have

(61) $$E = U_1 e^{-2\int V'}$$

Since $$\log\frac{\sqrt{E}}{R_2} = \log V,$$

the integration of the equation (57) give

$$\log V' + U = \log h_{02};$$

Consequently

(62) $$h_{02} = U' V.$$

We may write

(63) $$h_{01} = \frac{\sqrt{EG}}{R_1 R_2} = V'\frac{\sqrt{E}}{R_1}.$$

Equation (60) can now be put

in the form

$$\frac{\partial^2}{\partial v\,\partial s}\log\Pi + \frac{\partial^2}{\partial v\,\partial v}\log V' + \frac{\partial^2}{\partial v\,\partial v}\log\sqrt{E} - \frac{\partial^2}{\partial v\,\partial v}\log R_1 = 0.$$

But from (5-8) we see that the second and third terms of this equation vanish. It then becomes

$$\frac{\partial^2}{\partial v\,\partial v}\log\frac{\sqrt{E}}{R_1} = 0.$$

It is evident then from (39) that

$$h_{02} = \frac{\sqrt{E}\,\Pi}{R_1 R_2},$$

and consequently

$$h_{02} = h_{01}.$$

Combining this equation with (53), we get

$$h_{02} = V'\frac{\sqrt{E}}{R_1} = T'V';$$

therefore,

$$\frac{\sqrt{E}}{R_1} = T'.$$

From (2.3) we see then that

$$-\frac{\partial}{\partial u}\log\sqrt{E} = T'$$

The integration of this equation will give

$$k_2 = \nabla k^2 + J^2. \tag{65}$$

Since $k_{02} = k_{01} = J^2 V$,

we obtain by differentiation

$$\frac{\partial^2 \log k_{01}}{\partial v^2} = 0.$$

But it has been found that

$$\frac{\partial^2}{\partial v^2} \log \frac{\sqrt{E}}{R_1} = 0;$$

and by substituting this in the preceding equation we find

$$\frac{\partial^2}{\partial v^2} \log \frac{\sqrt{E}}{R_2} = 0.$$

This value substituted in the second of equations (32) will reduce it to the form

$$k_{01} = \frac{\sqrt{E}\, J}{R_1 R_2}.$$

Similarly we have then

$$k_{01} = k_{02} = k_{11} = k_{12}, \tag{66}$$

that is, all the invariants in

the two leading equations are now equal. The linear element in this case will be expressed by the equation

(67) $ds^2 = e^{-2(\sigma+\tau)}[\Gamma_1 e^{2\sigma} du^2 + \Gamma_2 e^{2\tau} dv^2]$.

If then u and v are the parameters of the lines of curvature of a surface and if the two first derived equations in the two series of Laplace (3) are identical, that is, $(E_{11}) = (E_{22})$; all the invariants in the two leading equations become equal to each other and the lines of curvature form an isothermal system.

6. We shall now consider

the general case and write $(E_{i,1}) = (E_{i,2})$. To do this it is evident that we must have the coefficients respectively identical in the two equations.

We have as in (44)

$$a_{i,1} = a_{0,1} - \frac{\partial}{\partial v} \log h_{0,1} h_{1,1} \cdots h_{i-1,1},$$

$$a_{i,2} = a_{0,2} - \frac{\partial}{\partial v} \log h_{0,2} h_{1,2} \cdots h_{i-1,2}$$

$$= a_{0,2} - \frac{\partial}{\partial v} \log h_{1,1} h_{2,1} \cdots h_{i,1}.$$

Equating these values of the coefficients, we get the equation

$$(65) \qquad a_{0,1} - \frac{\partial \log h_{0,1}}{\partial v} = a_{0,2} - \frac{\partial \log h_{i,1}}{\partial v}.$$

But $$a_{0,1} = -\frac{\sqrt{E}}{R_2},$$

and $$a_{0,2} = -\left(\frac{\partial}{\partial v} \log \frac{E}{R_1} + \frac{\sqrt{E}}{R_1}\right).$$

These values substituted in (65) reduce it to the form

$$\frac{\partial}{\partial v} \log h_{0,1} = \frac{\partial}{\partial v} \log \frac{\sqrt{E}}{R_1} + \frac{\partial}{\partial v} \log h_{i,1},$$

therefore, since

$$k_{o1} = \frac{\sqrt{\epsilon_0 \epsilon_4}}{R_1 R_2},$$

we obtain the equation

$$(69) \qquad \frac{\partial}{\partial v} \log \frac{\sqrt{\epsilon_4}}{R_2} = \frac{\partial}{\partial v} \log k_{i1}.$$

This is the first equation of condition for the identity proposed.

We know that

$$k_{i1} = k_{o1}, \quad k_{i2} = k_{o2};$$

then making

$$k_{i1} = k_{i2}$$

gives us simply

$$k_{o1} = k_{o2}.$$

Substituting the values of these coefficients the equation becomes

$$(70) \qquad \frac{\partial}{\partial v} \log \frac{\sqrt{\epsilon_4}}{R_2} = 0,$$

which is our second equation of condition.

By linear combinations

of the last of equations (72) we may obtain the general coefficient

$$(71)\quad c_{i,1} = c_{0,1} - \frac{\partial a_{0,1}}{\partial u} - \cdots - \frac{\partial a_{i-1,1}}{\partial u} + \frac{\partial b_{0,1}}{\partial v} + \cdots + \frac{\partial b_{i-1,1}}{\partial v} - b_{0,1}\frac{\partial \log h_{0,1}}{\partial v} \cdots h_{i-1,1}$$

Combinations of the first of equations (71) will give

$$(72)\quad -\frac{\partial a_{0,1}}{\partial u} - \cdots - \frac{\partial a_{i-1}}{\partial u} = -\frac{\partial^2}{\partial u \partial v}\log G^{\frac{i}{2}} + \frac{\partial}{\partial v}\log h_{0,1}^{i-1} h_{1,1}^{i-2} \cdots h_{i-2,1},$$

since, from (23), we have

$$\frac{\sqrt{G}}{R_1} = -\frac{\partial}{\partial v}\log\sqrt{G}.$$

It may readily be seen also that

$$(73)\quad \frac{\partial b_{0,1}}{\partial v} + \cdots + \frac{\partial b_{i-1,1}}{\partial v} = -\frac{\partial^2}{\partial v \partial u}\log\frac{h_2^{\frac{i}{2}}}{R_2^i} + \frac{\partial^2}{\partial v \partial u}\log h_2^{\frac{i}{2}}$$

$$= -\frac{\partial^2}{\partial u \partial v}\log R_2^{-i},$$

by remembering that

$$\frac{\sqrt{E}}{R_1} = -\frac{\partial}{\partial u}\log\sqrt{h_2}.$$

If these values in (72) and (73) be

substituted now in equation (71), it may be put in the form

$$c_{i1} = c_{01} - \frac{\partial^2}{\partial u \partial v} \log \frac{\Sigma_2^i}{R_2^i} + \frac{\partial^2}{\partial u \partial v} \log h_{01}^{i-1} h_{11}^{i-2} \cdots h_{i-1,1} - h_{01} \frac{\partial \log h_{01} \cdots h_{i-1,1}}{\partial v}$$

$$(74) \quad = c_{01} - \frac{\partial^2}{\partial u \partial v} \log \frac{\Sigma_2}{R_2} + \frac{\partial^2}{\partial u \partial v} \log \frac{\Sigma_2^{i-1}}{R_1^{i-1}} h_{11}^{i-2} \cdots h_{i-2,1} - h_{01} \frac{\partial \log h_{01} h_{11} \cdots h_{i-1,1}}{\partial v}.$$

But it may be seen from the first of equations (64) that proper linear combinations will give

$$(75) \quad h_{i-1,1} = \frac{\sqrt{\Sigma_2}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log \frac{\Sigma_2^{i-1}}{R_2^i} h_{11}^{i-2} h_{21} \cdots h_{i-2,1},$$

or

$$\frac{\partial^2}{\partial u \partial v} \log \frac{\Sigma_2^{i-1}}{R_1^i} h_{11}^{i-2} h_{21} \cdots h_{i-2,1} = \frac{\sqrt{\Sigma_2}}{R_1 R_2} - h_{i-1,1}.$$

If we substitute this value in equation (74) we have finally

$$(76) \quad c_{i1} = c_{01} - h_{01} - h_{i-1,1} - h_{01} \frac{\partial \log h_{01} h_{11} \cdots h_{i-1,1}}{\partial v}.$$

In the same manner we

may be shown that in the second series of equations (84) the general coefficient is

$$c_{i2} = c_{02} + k_{02} - k_{i-1,2} - b_{02}\frac{\partial \log f_{02} f_{12} \cdots f_{i-1,2}}{\partial v}$$

(87) $$= c_{02} + k_{01} - k_{i1} - b_{02}\frac{\partial \log f_{01} f_{11} \cdots f_{i1}}{\partial v},$$

by remembering the relations which exist between the invariants of the two series.

Since $k_{01} = k_{02}$,

$$c_{01} = c_{02} = 0,$$

if we place

$$c_{i1} = c_{i2},$$

we will get after obvious reductions the following equation

(78) $$k_{01} - k_{i-1,1} - b_{01}\frac{\partial \log f_{01}}{\partial v} = k_{01} - k_{i1} - b_{01}\frac{\partial \log f_{i1}}{\partial v}.$$

But we know that

$$k_{01} - k_{01} = -\frac{\lambda}{\sqrt{\theta}}\log\frac{\sqrt{\theta}}{r_2},$$

and from (34)

$$h_{i,1} - h_{i-1,1} = -\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} h_{1,1} h_{2,1} \cdots h_{i-1,1}.$$

Putting these values in equation (78), it may be written

$$-\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_2} - \frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{E}}{R_1} h_{1,1} h_{2,1} \cdots h_{i-1,1}$$

(141)

$$- h_{0,1} \frac{\partial \log h_{0,1}}{\partial v} + h_{0,1} \frac{\partial \log h_{0,1}}{\partial v} = 0.$$

The second of the second equation of condition (70) we have

$$h_{0,1} = -\frac{\sqrt{E}}{R_1},$$

and for the first condition for the preferred identity we found

$$\frac{\partial}{\partial v} \log h_{i,1} = \frac{\partial}{\partial v} \log \frac{\sqrt{E}}{R_2}.$$

Introducing these values in equation 79 above and making slight reductions, it becomes

$$-\frac{\partial^2}{\partial u \partial v} \log h_{0,1} h_{1,1} \cdots h_{i-1,1} + \frac{\partial}{\partial v} \frac{\sqrt{G}}{R_1} = 0.$$

Since $\frac{\sqrt{E_2}}{R_2} = -\frac{2}{\sqrt{2}} \log \sqrt{T_2}$

the last equation may now be written in the form

(F) $\frac{2^n}{n!} \log \sqrt{T} \, (c_1, c_2, \ldots c_{i-1}, c_i)$

which results from making the coefficients c_{i-1} and c_i identical.

The integration of the second equation of condition (70) will give

$$\log \frac{\sqrt{E}}{R_2} = \log T.$$

or

$$\frac{\sqrt{E_2}}{R_2} = T.$$

But from (2) we have

$$\frac{\sqrt{E_2}}{R_2} = -\frac{2}{\sqrt{2}} \log \sqrt{\theta},$$

hence

$$-\frac{2}{\sqrt{2}} \log \sqrt{\theta} = T.$$

By integrating the last equation we get

(G) $$\theta = U_0 e^{-\sqrt{2}T}.$$

The equation

$$\frac{\partial}{\partial v}\log k_{i,1} = \frac{\partial}{\partial v}\log\frac{\sqrt{E_2}}{R_2}$$

may be integrated and placed in the form

$$\log k_{i,1} = \log\frac{\sqrt{E_2}}{R_2} + \log S' = \log T - \log T_1'$$

therefore

$$k_{i,1} = T' T_1' \tag{8a}$$

Equation (8) may be written as follows:

$$\left(\frac{\partial^2}{\partial u \partial v}\log\sqrt{E_1} + \frac{\partial^2}{\partial u \partial v}\log\sqrt{E_2} - \frac{\partial^2}{\partial u \partial v}\log R_1\right.$$

$$\left. + \frac{\partial^2}{\partial u \partial v}\log\frac{\sqrt{E_2}}{R_2} + \frac{\partial^2}{\partial u \partial v}\log k_{1,1} k_{2,1} \cdots k_{i-1,1} = 0.\right.$$

But it has previously been shown that the second and fourth terms of this equation are equal to zero. It then becomes

$$\frac{\partial^2}{\partial u \partial v}\log\frac{\sqrt{E_2}}{R_2} + \frac{\partial^2}{\partial u \partial v}\log k_{1,1} k_{2,1} \cdots k_{i-1,1} = 0 \tag{13}$$

From (3a) we know that

$$k_{i,1} = k_{i-1,1} - \frac{\partial^2}{\partial \alpha_i \partial \alpha_1} \log \frac{\sqrt{E}}{E_1} - \frac{\partial^2}{\partial \alpha_i \partial \alpha_1} \log k_{i,1} \dots k_{i-1,1}.$$

Hence by combining this with the equation above, we get

$$k_{i,1} = k_{i-1,1}.$$

But we have found

$$k_{2,1} = F' E',$$

consequently

$$k_{i-1,1} = F' E'. \tag{84}$$

The differentiation of the last equation will yield

$$\frac{\partial^2}{\partial \alpha_i \partial \alpha_1} \log k_{i-1,1} = 0 \tag{85}$$

and if this be substituted in (83), we have

$$\frac{\partial^2}{\partial \alpha_i \partial \alpha_1} \log \frac{\sqrt{E}}{E_1} + \frac{\partial^2}{\partial \alpha_i \partial \alpha_1} \log k_{i,1} k_{i,1} \dots k_{i-2,1} = 0.$$

Again $k_{i-1,1} = k_{i-2,1} - \frac{\partial^2}{\partial \alpha_i \partial \alpha_1} \log \frac{\sqrt{E}}{E_1} - \frac{\partial^2}{\partial \alpha_i \partial \alpha_1} \log k_{i,1} \dots k_{i-2,1}$

and, consequently

$$E_{i-1,1} = h_{i-2,1}.$$

Comparing this with equation (85), we have

$$\frac{\partial^2}{\partial u \partial v} \log h_{i-2,1} = 0.$$

Substituting, as before, (F) now becomes

$$(87) \quad \frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} + \frac{\partial^2}{\partial u \partial v} \log h_{11}, h_{21}, \cdots h_{i-3,1} = 0$$

Thus, it may easily be seen that by continuing this process, each time, we are left finally with the equation

$$\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{E_1} = 0.$$

Then we have

$$h_{11} = h_{01} = h_{02}.$$

If we should make the equations (E_{01}) and (E_{11}) identical

it would give

$$h_{01} = h_{11}, \quad k_{01} = k_{11}.$$

But from the relations of the invariants in the two series we have

$$h_{11} = h_{02}, \quad k_{11} = k_{02};$$

hence

$$h_{02} = h_{01} = h_{11},$$
$$k_{02} = k_{01} = k_{11}$$

Then, by making the equations (Θ_{i1}) and (Θ_{i2}) identical, we find that all the invariants in the two series are equal and we are led only to the consideration of

$$\Theta_{01} = \Theta_{11}.$$

Since

$$(\text{if}) \quad \begin{cases} h_{01} = h_{02}, \\ k_{01} = k_{02}, \end{cases}$$

we have

$$\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{E_0}}{R_0} = 0$$

$$\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{E_0}}{R_{12}} = 0.$$

The integrals of these two equations may be written in the form

$$\frac{\sqrt{E_1}}{R_1} = U_2 V_1,$$

$$\frac{\sqrt{E_2}}{R_2} = U_1 V_2.$$

Substituting the values of $\frac{\sqrt{E_1}}{R_1}$ and $\frac{\sqrt{E_2}}{R_2}$ from equations (21), we get

$$-\frac{1}{2\sqrt{E_1 G_1}} \frac{\partial G_1}{\partial u} = U_2 V_1,$$

$$-\frac{1}{2\sqrt{E_2 G_2}} \frac{\partial E_2}{\partial v} = U_1 V_2.$$

The elimination of $-\frac{1}{2\sqrt{E G}}$ from these equations will give

$$U_2 V_1 \frac{\partial E_2}{\partial v} = U_1 V_2 \frac{\partial G_1}{\partial u},$$

or

$$\frac{U_2}{U_1} \frac{\partial E_2}{\partial v} = \frac{V_2}{V_1} \frac{\partial G_1}{\partial u}$$

where the U's are functions of u alone and the V's of v alone.

Write now

$$U_2 = U U_1,$$
$$V_2 = V V_1;$$

than the equation above becomes

$$\frac{\partial}{\partial s}(T\bar{e}) = \frac{\partial}{\partial n}(Tf). \tag{29}$$

Next there is a necessary condition for the existence of an exact differential equation

$$(Te)\,dx + (Tf)\,dv = dq, \tag{9}$$

where

$$\bar{e} = \frac{1}{T}\frac{\partial p}{\partial x}$$
$$f = \frac{1}{T}\frac{\partial p}{\partial v}.$$

Take u and v arbitrary and p will have to be determined.

From (23) we have

$$\frac{\sqrt{E}}{R_1} = -\frac{\partial}{\partial v}\log\sqrt{E},$$

and by introducing the value of E from the first of the two equations above, it becomes

$$\frac{\sqrt{E}}{R_1} = -\frac{\partial}{\partial v}\left(-\frac{1}{2}\log T + \frac{1}{2}\log\frac{\partial p}{\partial u}\right),$$

Therefore,

$$\frac{\sqrt{G}}{R_2} = -\frac{1}{2}\frac{\partial}{\partial v}\log\frac{\partial\varphi}{\partial u} = -\frac{1}{2}\frac{\frac{\partial^2\varphi}{\partial u\,\partial v}}{\frac{\partial\varphi}{\partial u}}.$$

In like manner we can find

$$\frac{\sqrt{E}}{R_1} = -\frac{1}{2}\frac{\frac{\partial^2\varphi}{\partial u\,\partial v}}{\frac{\partial\varphi}{\partial v}}.$$

The product of these two expressions will give

$$\frac{\sqrt{EG}}{R_1 R_2} = \frac{1}{4}\frac{\left(\frac{\partial^2\varphi}{\partial u\,\partial v}\right)^2}{\frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}} = \sigma T_1^2 \nabla V_1^2. \tag{91}$$

If we put

$$\nabla V = e^{\frac{1}{2}\int \sigma T_1^2 du + c}\,\nabla V_1^2 du,$$

we have then

$$\frac{\partial^2 \log \nabla V}{\partial u^2} = 2\sigma T_1^2,$$

$$\frac{\partial^2 \log \nabla V}{\partial v^2} = 2\nabla V_1^2.$$

Hence, equation (91) may now be placed in the form

$$\frac{\left(\frac{\partial^2\varphi}{\partial u\,\partial v}\right)^2}{\frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}} = \frac{\partial^2 \log \nabla V}{\partial u^2}\cdot\frac{\partial^2 \log \nabla V}{\partial v^2}. \tag{92}$$

7. We propose now to consider the result of making $k_{01} = a_{01} = 0$, that is

$$\int \frac{2}{\partial v} \log \frac{\sqrt{E}}{R_2} + \frac{\sqrt{E}}{R_{21}} = 0,$$

$$\int \frac{2}{\partial v} \log \frac{\sqrt{G}}{R_{21}} + \frac{\sqrt{E}}{R_2} = 0. \tag{93}$$

By substituting the values found in (24) the two equations above readily reduce to

$$\frac{2}{\partial u} \log R_2 = 0,$$

$$\frac{2}{\partial v} \log R_1 = 0.$$

The integrals of these equations are

$$\begin{cases} R_2 = f(v), \\ R_1 = \varphi(u). \end{cases} \tag{94}$$

We may, therefore, write

$$-\frac{1}{R_1} = \frac{1}{\sqrt{EG}} \frac{\partial \sqrt{E}}{\partial u} = T_1$$

$$-\frac{1}{R_2} = \frac{1}{\sqrt{EG}} \frac{\partial \sqrt{G}}{\partial v} = T_2$$

where T is a function only of u and ∇ a function of v only. By eliminating $\sqrt{E}\sqrt{G}$ we obtain the equation

$$\nabla \frac{\partial \sqrt{T}}{\partial u} = T \frac{\partial \sqrt{\Theta}}{\partial v},$$

or

$$\frac{\partial}{T\,\partial u} T\sqrt{T} = \frac{\partial}{\partial v} T\sqrt{\Theta}.$$

We see that this is a condition for the exact differential equation

$$(T\sqrt{\Theta})\,du + (T\sqrt{T})\,dv = d\varphi,$$

in case we have

$$\sqrt{E} = \frac{1}{T}\frac{\partial \varphi}{\partial u},$$

$$\sqrt{T} = \frac{1}{T}\frac{\partial \varphi}{\partial v}.$$

We find as before

$$\frac{\sqrt{E}}{\sqrt{G}} = -\frac{\dfrac{\partial^2 \varphi}{\partial u\,\partial v}}{\dfrac{\partial \varphi}{\partial u}},$$

$$\frac{\sqrt{G}}{\sqrt{E}} = -\frac{\dfrac{\partial^2 \varphi}{\partial u\,\partial v}}{\dfrac{\partial \varphi}{\partial v}},$$

and, therefore, the differential equation

$$\frac{\partial^2 \omega}{\partial \omega^2} = \frac{\partial \omega}{\partial \omega} \frac{\partial \omega}{\partial \omega}. \qquad (97)$$

The integral of this may be written

$$\lambda = -\log(T_1 + T_1'). \qquad (98)$$

It may then easily be seen that

$$\frac{E}{c^2} = \frac{T_1^2}{T^2(T_1 + T_1')^2 - 1}$$

$$\frac{1}{\lambda} = \frac{T_1^2}{T^2(T_1 + T_1')^2}.$$

The linear element may now be expressed by the equation

$$ds^2 = \frac{1}{(T_1 + T_1')^2}\left[\frac{T_1^2}{U^2}du^2 + \frac{T_1'^2}{V^2}dv^2\right]. \qquad (99)$$

or $$ds^2 = \frac{1}{\lambda}(U du^2 + V dv^2).$$

This is a form which characterizes isothermal systems.

That λ is a function of u and v but the product of a function of u alone and a function of v alone

a surface, and if the radius of geodesic curvature of the line $v = \text{const.}$ on a function of u alone and that of $u = \text{const.}$ of v alone, the lines of curvature of the surface will form an isothermal system.

The leading equations of the two series now become

$$(E_0)_1 = \frac{\partial^2 x_1}{\partial u^2} - \frac{\sqrt{E}}{R_1}\frac{\partial p_1}{\partial u} = 0,$$

$$(E_0)_2 = \frac{\partial^2 x_2}{\partial u^2} - \frac{\sqrt{G}}{R_1}\frac{\partial x_2}{\partial v} = 0,$$

while that satisfied by x, y, z becomes

$$\text{or} \qquad F_2 = \frac{\partial^2 x}{\partial u^2} + \frac{\sqrt{E}}{R_1}\frac{\partial x}{\partial u} + \frac{\sqrt{G}}{R_2}\frac{\partial x}{\partial v} = 0.$$

Since $\sqrt{T} = T_1 + T_2$

we obtain

$$\frac{\sqrt{E}}{R_1} = -\frac{\partial \log \sqrt{E}}{\partial u} = \frac{T_1'}{T_1 + T_2} = \frac{\partial \log \sqrt{T}}{\partial u}.$$

$$\frac{\sqrt{G}}{R_1} = -\frac{\partial \log \sqrt{\lambda}}{\partial u} = \frac{U_1'}{U_1 - V_1} = -\frac{\partial \log \sqrt{\lambda}}{\partial v}.$$

By substituting these in the equations above we get

$$G_{01} = \frac{\partial^2 \varphi_{01}}{\partial u \partial v} - \frac{\partial \log \sqrt{\lambda}}{\partial v}\frac{\partial \varphi_{01}}{\partial u} = 0,$$

$$G_{02} = \frac{\partial^2 \varphi_{02}}{\partial u \partial v} - \frac{\partial \log \sqrt{\lambda}}{\partial u}\frac{\partial \varphi_{02}}{\partial v} = 0,$$

$$G = \frac{\partial^2 \varphi}{\partial u \partial v} - \frac{\partial \log \sqrt{\lambda}}{\partial v}\frac{\partial \varphi}{\partial u} - \frac{\partial \log \sqrt{\lambda}}{\partial u}\frac{\partial \varphi}{\partial v} = 0$$

The integrals of the three preceding equations are now

$$h_{01} = -\frac{\partial}{\partial u}\log\sqrt{\lambda},$$

$$k_{01} = 0,$$

$$k_{02} = 0,$$

$$h_{02} = -\frac{\partial}{\partial v}\log\sqrt{\lambda}.$$

That is, they have their increments equal but taken in different order.

8. Next we shall consider the result of making the coefficients of the equations (6_{01}) and (6_{02}) equal but taken in opposite order, that is

$$x_{01} = b_{02}, \quad x_{02} = b_{01}.$$

By substituting their values we have

$$\frac{\sqrt{E_1}}{R_2} = \frac{\sqrt{E_2}}{R_1},$$

$$\frac{\partial}{\partial u}\log\frac{\sqrt{E_1}}{R_2} + \frac{\sqrt{E_2}}{R_1} = \frac{\partial}{\partial v}\log\frac{\sqrt{E_2}}{R_1} + \frac{\sqrt{E_1}}{R_2}.$$

By combining these we get

$$\frac{\partial}{\partial u}\log\frac{\sqrt{E_1}}{R_2} = \frac{\partial}{\partial v}\log\frac{\sqrt{E_2}}{R_1},$$

or

$$\frac{R_2}{\sqrt{E_1}}\cdot\frac{\partial}{\partial u}\cdot\frac{\sqrt{E_1}}{R_2} = \frac{R_1}{\sqrt{E_2}}\cdot\frac{\partial}{\partial v}\cdot\frac{\sqrt{E_2}}{R_1}.$$

Taking into account the first of equations (16) we have

$$(17) \qquad \frac{\partial}{\partial u}\frac{\sqrt{E}}{R_2} = \frac{\partial}{\partial v}\frac{\sqrt{E_2}}{R_1}.$$

But $\frac{\sqrt{G}}{R_2} = -\frac{\partial^2}{\partial v^2} \log \sqrt{E}$.

and $\frac{\sqrt{E}}{R_1} = -\frac{\partial^2}{\partial u^2} \log \sqrt{G}$.

These values placed in equation (107) will give

$$\frac{\partial^2}{\partial u \partial v} \log \sqrt{E} = \frac{\partial^2}{\partial u \partial v} \log \sqrt{G}.$$

If we integrate this equation we shall get

$$\sqrt{E} = \phi_1 U V_1 \qquad (108)$$

where U is a function of u only and V of v only. The linear element can now be put in the form

$$ds^2 = \phi_1 (U V du^2 + dv^2),$$

or

$$ds^2 = \phi_1 V (U du^2 + \frac{1}{V} dv^2). \qquad (109)$$

Thus the lines of curvature form an isothermal system.

9. Let us now consider the forms to which the equations (E_{0_1} and E_{0_2}) reduce for surfaces parallel to the original one.

Since the surface is referred to its lines of curvature we have the following values for parallel surfaces:

$$(110)\qquad \begin{cases} E_0 = E\left(1 - \dfrac{a}{\rho_1}\right)^2 = E\left(\dfrac{\rho_1^0}{\rho_1}\right)^2, \\ G_0 = G\left(1 - \dfrac{a}{\rho_2}\right)^2 = G\left(\dfrac{\rho_2^0}{\rho_2}\right)^2, \end{cases}$$

$$(111)\qquad \begin{cases} \rho_1^0 = \rho_1 - a, \\ \rho_2^0 = \rho_2 - a, \end{cases}$$

where a is a constant which changes as we pass from one surface to the next one.

We have now

$$(112_3)\quad \begin{cases} \dfrac{\partial}{\partial u}\dfrac{1}{\rho_1} = \left(\dfrac{\rho_1^0}{\rho_1^0+u_1}\right)^2 \dfrac{\partial}{\partial u}\dfrac{1}{\rho_1^0}, \\ \dfrac{\partial}{\partial v}\dfrac{1}{\rho_1} = \left(\dfrac{\rho_1^0}{\rho_1^0+u_1}\right)^2 \dfrac{\partial}{\partial v}\dfrac{1}{\rho_1^0}. \end{cases}$$

By differentiating the first of these equations for v and the second for u we get

$$(113)\quad \begin{cases} \dfrac{\partial^2}{\partial u \partial v}\dfrac{1}{\rho_1} = \left(\dfrac{\rho_1^0}{\rho_1^0+u_1}\right)^2 \dfrac{\partial^2}{\partial u \partial v}\dfrac{1}{\rho_1^0} + \dfrac{\partial}{\partial v}\left(\dfrac{\rho_1^0}{\rho_1^0+u_1}\right)^2 \dfrac{\partial}{\partial u}\dfrac{1}{\rho_1^0}, \\ \dfrac{\partial^2}{\partial u \partial v}\dfrac{1}{\rho_1} = \left(\dfrac{\rho_1^0}{\rho_1^0+u_1}\right)^2 \dfrac{\partial^2}{\partial u \partial v}\dfrac{1}{\rho_1^0} + \dfrac{\partial}{\partial u}\left(\dfrac{\rho_1^0}{\rho_1^0+u_1}\right)^2 \dfrac{\partial}{\partial v}\dfrac{1}{\rho_1^0}. \end{cases}$$

By substituting these values from (112) and (113) the equation

$$\frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_1} - \frac{\sqrt{E_2}}{R_2}\frac{\partial}{\partial u}\frac{1}{\rho_1} - \left(\frac{\partial}{\partial u}\log\frac{\sqrt{E_1}}{R_2} + \frac{\sqrt{G}}{E_1}\frac{\partial}{\partial v}\right)\frac{1}{\rho_1} = 0$$

can be put in either of the two following forms:

$$(114)\quad \begin{cases} \dfrac{\partial^2}{\partial u \partial v}\dfrac{1}{\rho_1^0} - \left[\dfrac{\sqrt{E_2}}{R_2} - \dfrac{\partial}{\partial v}\log\dfrac{\rho_1^0}{\rho_1^0+u_1}\right]\dfrac{\partial}{\partial u}\dfrac{1}{\rho_1^0} \\ \qquad - \left(\dfrac{\partial}{\partial u}\log\dfrac{\sqrt{E_2}}{R_2} + \dfrac{\sqrt{G}}{E_1}\right)\dfrac{\partial}{\partial v}\dfrac{1}{\rho_1^0} = 0, \\ \dfrac{\partial^2}{\partial u \partial v}\dfrac{1}{\rho_1^0} - \dfrac{\sqrt{E_2}}{R_2}\dfrac{\partial}{\partial u}\dfrac{1}{\rho_1^0} \\ \qquad - \left[\dfrac{\partial}{\partial u}\log\dfrac{\sqrt{E_2}}{R_2} + \dfrac{\sqrt{G}}{E_1} - \dfrac{\partial}{\partial u}\log\left(\dfrac{\rho_1^0}{\rho_1^0+u_1}\right)^2\right]\dfrac{\partial}{\partial v}\dfrac{1}{\rho_1^0} = 0. \end{cases}$$

By examining these we see that in order that these equations reduce to the same form as the original equation, we must have

$$(115)\quad \begin{cases} \frac{\partial}{\partial r}\log\left(\frac{\rho_1^0}{\rho_1^{0+2}}\right)^{-} = 0, \\ \frac{\partial}{\partial u}\log\left(\frac{\rho_2^0}{\rho_2^{0+2}}\right)^{-} = 0. \end{cases}$$

The integration of these will give

$$(116)\quad \begin{cases} \rho_1^0 = \rho_1 u, \\ \rho_2^0 = \rho_2 v. \end{cases}$$

In the same manner as above the equation (E_{0u}) may be put in either of the forms

$$\text{or} \begin{cases} \frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_2^0} - \frac{\partial}{\partial v}\log\frac{\sqrt{E_0}}{R_1} + \frac{\sqrt{E}}{R_2} - \frac{\partial}{\partial v}\log\left(\frac{\rho_2^0}{\rho_2^{0+2}}\right)^{-}\Big]\frac{\partial}{\partial u}\frac{1}{\rho_2^0} \\ \qquad - \frac{\sqrt{E_0}}{R_1}\frac{\partial}{\partial v}\frac{1}{\rho_2^0} = 0, \\ \frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_2^0} - \left[\frac{\partial}{\partial v}\log\frac{\sqrt{E_0}}{R_1} + \frac{\sqrt{E}}{R_2}\right]\frac{\partial}{\partial u}\frac{1}{\rho_2^0} \\ \qquad - \left[\frac{\sqrt{E_0}}{R_1} - \frac{\partial}{\partial u}\log\left(\frac{\rho_2^0}{\rho_2^{0+2}}\right)^{-}\right]\frac{\partial}{\partial v}\frac{1}{\rho_2^0} = 0. \end{cases}$$

The condition that these reduce to the Monna form $\varphi(E,\eta)$ is, as before,

$$(118)\qquad \begin{cases} \rho_2^0 = \varphi(\eta), \\ \rho_2^0 = \rho_1(v). \end{cases}$$

Then if each of the principal radii of curvature of the parallel surfaces is a function of only one of the parameters v and η, the reciprocals of those radii will satisfy the Monna partial differential equations (E_0,η) and (E_0,v), as the reciprocals of the principal radii of curvature of the original surface.

From (110) we have

$$\frac{\sqrt{E}}{\rho_1} = \frac{\sqrt{E_0}}{\rho_1^0},$$

$$\frac{\sqrt{\eta}}{\rho_2} = \frac{\sqrt{\eta_0}}{\rho_2^0}.$$

Then, the equations

$$\frac{\partial^2}{\partial u\,\partial v}\frac{\sqrt{G}}{R} - \frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2}\frac{\partial}{\partial v}\frac{\sqrt{E}}{R} - \frac{\sqrt{G}\sqrt{E}}{R_1 R_2}\frac{\sqrt{E}}{R} = 0$$

$$\frac{\partial^2}{\partial u\,\partial v}\frac{\sqrt{E}}{R} - \frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1}\frac{\partial}{\partial u}\frac{\sqrt{E}}{R} - \frac{\sqrt{G}\sqrt{E}}{R_1 R_2}\frac{\sqrt{E}}{R} = 0$$

by a change from u, v to u, v when we pass to a parallel surface.

10. (a) For the ellipsoid we have

$$E = \frac{u(u-v)}{f(u)}, \quad G = \frac{v(v-u)}{f(v)}.$$

Then the equations E_{01} and E_{02} become

$$(119)\quad \begin{cases} \dfrac{\partial^2\varphi_{01}}{\partial u\,\partial v} - \dfrac{1}{2(u-v)}\dfrac{\partial\varphi_{01}}{\partial u} + \dfrac{3}{2(u-v)}\dfrac{\partial\varphi_{01}}{\partial v} = 0, \\ \dfrac{\partial^2\varphi_{02}}{\partial u\,\partial v} - \dfrac{3}{2(u-v)}\dfrac{\partial\varphi_{02}}{\partial u} + \dfrac{1}{2(u-v)}\dfrac{\partial\varphi_{02}}{\partial v} = 0; \end{cases}$$

and the equation satisfied by (x, y, z) is

$$(120)\quad \frac{\partial^2\varphi_0}{\partial u\,\partial v} + \frac{1}{2(u-v)}\frac{\partial\varphi_0}{\partial u} - \frac{1}{2(u-v)}\frac{\partial\varphi_0}{\partial v} = 0.$$

We can obtain a great number of particular integrals of

these equations. For example, we can find solutions which are the product of a function of u by a function of v.

If a denotes a "constant, we write

$$\varphi_{01} = (u-a)^{-\frac{3}{2}}(v-a)^{-\frac{1}{2}} = \frac{1}{\sqrt{(u-a)^3(v-a)}},$$

$$\varphi_{02} = (u-a)^{-\frac{1}{2}}(v-a)^{-\frac{3}{2}} = \frac{1}{\sqrt{(u-a)(v-a)^3}}.$$

By using a known property of these equations we have also

$$\varphi_{01} = (v-u)^{-1}(u-a)^{-\frac{1}{2}}(v-a)^{\frac{1}{2}} = \frac{1}{v-u}\sqrt{\frac{v-a}{u-a}},$$

$$\varphi_{02} = (v-u)^{-1}(u-a)^{\frac{1}{2}}(v-a)^{-\frac{1}{2}} = \frac{1}{v-u}\sqrt{\frac{u-a}{v-a}}.$$

We can find new solutions by operating on these already found with the symbols

$$\frac{\partial}{\partial u} \text{ or } \frac{\partial}{\partial v},$$

$$u\frac{\partial v}{\partial u} + v\frac{\partial v}{\partial v},$$

$$u^2\frac{\partial}{\partial u} + v^2\frac{\partial}{\partial v} + \frac{3}{2}u + \frac{1}{2}v.$$

We have also the general formulæ

$$P_{0,1} = \int_u^v f(x)(x-u)^{-\frac{3}{2}}(v-x)^{-\frac{1}{2}}\,dx$$

$$+ (v-u)^{-1}\int_u^v f_1(x)(x-u)^{-\frac{1}{2}}(v-x)^{\frac{1}{2}}\,dx,$$

$$P_{0,2} = \int_u^v f(x)(x-u)^{-\frac{1}{2}}(v-x)^{-\frac{3}{2}}\,dx$$

$$+ (v-u)^{-1}\int_u^v f_1(x)(x-u)^{\frac{1}{2}}(v-x)^{-\frac{1}{2}}\,dx,$$

which contain two distinct arbitrary functions.

In the same way we can obtain integrals of equation (2).

The integration of this equation will determine the two forms which admit for adjoint representation of their lines of curvature a system of algebraic homofocal ellipses. If u and

ν denote the parameters of these ellipses and λ, μ, ν the coordinates of a point of the plane, we have

$$\lambda^2 = \frac{(a-\alpha)(a-\beta)}{(a-b)(a-c)},$$

$$\mu^2 = \frac{(b-\alpha)(b-\beta)}{(b-a)(b-c)},$$

$$\nu^2 = \frac{(c-\alpha)(c-\beta)}{(c-a)(c-b)},$$

and the values of λ, μ, ν will satisfy equation (1) $z=1$.

The solution

$$p_0 = C + C'(\lambda + \mu)$$

will give the surface of the funnel, whose shape is defined by the equation

$$p = \frac{a\lambda^2 + b\mu^2 + c\nu^2}{2}$$

where λ, μ, ν are the direction cosines of the normal, and the coordinates of the point of

contact of the tangent plane from the values

$$x = (\rho - a)\lambda, \quad y = (\rho - b)\mu, \quad z = (\rho - c)\nu.$$

If we take the solution

$$\rho_0 = c\sqrt{(u-a)(v-a)},$$

we obtain the surfaces of the second degree.

Equation (120) also admits as particular solutions the five confocal coordinates of the ellipsoid defined by the formula

$$x_i = \sqrt{\frac{(x_i - a)(a_i - u)(a_i - v)}{f'(a_i)}}, \quad i = 1, 2, 3, 4, 5$$

(b). For the ellipsoid of Lamé the coefficients of the linear element are

$$E = \frac{1}{\sqrt{f(u-v)}}, \quad G = \frac{1}{\sqrt{f(v-u)}}.$$

The equations E_{01} and E_{02} now become

$$(121)\quad \begin{cases} \dfrac{\partial^2 \varphi_{01}}{\partial u \partial v} + \dfrac{1}{u-v}\dfrac{\partial \varphi_{01}}{\partial u} = 0 \\ \dfrac{\partial^2 \varphi_{02}}{\partial u \partial v} - \dfrac{1}{u-v}\dfrac{\partial \varphi_{02}}{\partial v} = 0 \end{cases}$$

while that satisfied by the point coordinates is

$$(122)\quad E_0 = \frac{\partial^2 \varphi_0}{\partial u \partial v} - \frac{1}{u-v}\frac{\partial \varphi_0}{\partial u} + \frac{1}{u-v}\frac{\partial \varphi_0}{\partial v} = 0$$

We can obtain any number of algebraic integrals of these equations; for example,

$$\varphi_{01} = v - a \quad , \quad \varphi_{02} = u - a;$$

$$\varphi_{01} = \frac{(u-v)^2}{(v-a)(u-a)} \quad , \quad \varphi_{02} = \frac{(u-v)^2}{(u-a)(v-a)}.$$

The integration of (E_0) gives

$$\varphi_0 = \frac{T - V}{u - v}.$$

Now for φ_0, by making the substitution $\varphi_0 = \frac{\theta}{u-v}$, the equation E_0 is transformed into $\frac{\partial^2 \theta}{\partial u \partial v} = 0$

Life.

I was born at Loachapoka, Ala., Nov. 7, 1867, and received my early training at the High School at that place. I was graduated from the Southern University (Ala.) with the degree of Bachelor of Science in 1887. After spending one year as principal of a high school, I returned to my Alma Mater, where I remained two years as instructor in Mathematics, and received the degree of A.M. in 1890. Having spent another year teaching, I entered the Johns Hopkins University in 1891, where I pursued graduate

Courses in Mathematics, Physics, and Astronomy. During 1892-'96 I was Professor of Mathematics in Millsaps College (Miss.). In 1896 I returned to this University. Those who taught the teachers of Professors Craig and Franklin and Drs. Harris, Poe, Hilgard, and Shepherd, to all of whom I am grateful for the kindnesses that I have received from them.

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